Effects of Lognormal Amplitude Fading on Bit Error Probability for Uncoded Binary PSK Signaling

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The 1978 Pioneer Venus mission will require direct communication links between the planetary probes and Earth. Data from the Russian spacecraft Venera 4 indicate that these links will be subjected to lognormal fading resulting from atmospheric turbulence. This article analyzes the bit error rate degradation for uncoded binary phase-shift-keyed (PSK) telemetry in the presence of such fading.

I. Introduction

The 1978 Pioneer Venus mission will require direct communication links between the planetary probes and Earth. A review of the Russian Venera data indicates that these links will be subjected to lognormal fading due to the turbulent atmosphere of Venus (Ref. 1). This paper analyzes the degradation of the bit error rate for uncoded binary phase-shift-keyed (PSK) signals received over the additive white Gaussian noise (AWGN) channel in the presence of such fading.

II. Low and High Rate Bounds

Consider an uncoded binary PSK communication link over the AWGN channel. In the absence of fading, the received signal has the form

$$r(t) = \sqrt{2}A \cos[\omega_c t + \theta_m d(t)\, sq(\omega_s t)] + n(t) \qquad (1)$$

where ω_c is the carrier frequency, θ_m is the modulation index ($\theta_m < 90°$), $d(t)$ is the binary data with baud time T_B ($d(t) = \pm 1$), $sq(\omega_s t)$ is the squarewave subcarrier at frequency ω_s ($2\pi/T_B << \omega_s << \omega_c$), and $n(t)$ is a wideband Gaussian noise process. If the channel has atmospheric fading of the form anticipated for Pioneer Venus, the received signal will be (Refs. 2 and 3)

$$r(t) = \sqrt{2}A\, e^{\chi(t)} \cos[\omega_c t + \theta_m d(t)\, sq(\omega_s t) + \phi(t)] + n(t) \qquad (2)$$

where $\chi(t)$ and $\phi(t)$ are stationary, jointly Gaussian random processes, and $e^{\chi(t)}$ is a lognormal random process.

From Venera 4 data, Woo (Ref. 1) has concluded that $\chi(t)$ and $\phi(t)$ are narrowband processes, with one-sided power spectral bandwidths

$$W_\chi, W_\phi \sim 1\ \text{Hz} \qquad (3)$$

It is assumed that the phase fading process is sufficiently narrowband relative to the carrier phase-locked loop bandwidth in the receiver that it can be tracked without difficulty. Consequently, the analysis below neglects $\phi(t)$ and assumes that all of the degradation in link performance due to the fading is caused by the amplitude fading process $e^{\chi(t)}$. The bit error rate, conditioned on the fading, has the form (Ref. 4)

$$p(\epsilon \mid \alpha) = Q(\alpha\sqrt{2\rho}) \tag{4}$$

where

$$\alpha \equiv \frac{1}{T_B}\int_0^{T_B} dt\, e^{\chi(t)} \tag{5}$$

$$\rho \equiv \frac{A^2 \sin^2\theta_m T_B}{N_0} \tag{6}$$

$$Q(\zeta) \equiv \frac{1}{\sqrt{2\pi}}\int_\zeta^\infty dx \exp\left(-\frac{x^2}{2}\right) \tag{7}$$

Suppose the data rate R_B is high:

$$R_B \equiv \frac{1}{T_B} >> W_\chi \tag{8}$$

Then $\chi(t)$ is essentially constant over the baud time T_B, and α reduces to a lognormal random variable

$$\alpha = e^\chi \tag{9}$$

where $\chi \equiv \chi(t_0)$ for some $t_0 \epsilon (0, T_B)$, and its mean m_χ is the negative of its variance σ_χ^2 (Ref. 3). Then the expected bit error rate is given by

$$P(\epsilon) = \overline{Q(e^\chi\sqrt{2\rho})}^{\chi} \tag{10}$$

Now consider the extremely low data rate case, defined by

$$R_B << W_\chi \tag{11}$$

Then α is a long time average of $e^{\chi(t)}$, and assuming $\chi(t)$ is ergodic,

$$\alpha = \overline{e^\chi} = \exp\left(-\frac{\sigma_\chi^2}{2}\right) \tag{12}$$

Then

$$P(\epsilon) = Q\left[\sqrt{2\rho \exp\left[-\sigma_\chi^2\right]}\right] \tag{13}$$

Woo has computed a variance $\sigma_\chi^2 = 0.014$ for the Venusian atmosphere (Ref. 1). For this variance, Eqs. (10) and (13) are compared in Fig. 1 with the nonfading case ($\chi = 0$); the parameter β in Fig. 1 is defined by

$$\beta \equiv 2\pi W_\chi / R_B \tag{14}$$

The $\beta = 0$ curve corresponds to Eq. (10); it had to be computed numerically using a 20th-order Hermite integration formula (Ref. 5). This curve indicates a signal-to-noise ratio degradation of 0.4 dB at a bit error rate of 10^{-2}, and 0.9 dB at 10^{-4} due to the fading. By comparison, the infinite β curve of Eq. (13) has a degradation of

$$-10\log_{10}(\exp[-\sigma_\chi^2]) = 0.06 \text{ dB};\ \sigma_\chi^2 = 0.014 \tag{15}$$

independent of the bit error rate.

For intermediate data rates ($0 < \beta < \infty$), it can be shown that the bit error rate curve is bounded by the $\beta = 0$ and infinite β curves. Applying Eq. (A.4) derived in Appendix A, we have

$$Q\left(\sqrt{2\rho e^{-\sigma_\chi^2}}\right) \le P(\epsilon) \le \overline{Q(e^\chi\sqrt{2\rho})}^{\chi} \tag{16}$$

In particular, the $\beta = 0$ curve of Eq. (10) represents a worst-case fading degradation of the bit error rate over all data rates.

III. Intermediate Rate Model

The following analysis is adapted from the work of Tausworthe (Ref. 6) and Layland (Ref. 7) on noisy reference detection.

Suppose the covariance function of $\chi(t)$ can be approximated by the expression

$$\begin{aligned} R_\chi(\tau) &\equiv \overline{[\chi(t+\tau) - \overline{\chi(t+\tau)}]\,[\chi(t) - \overline{\chi(t)}]} \\ &\cong \sigma_\chi^2 \exp(-2\pi W_\chi|\tau|) \end{aligned} \tag{17}$$

Equation (17) satisfies the requirement that $R_\chi(0) = \sigma_\chi^2$. It also yields a power spectral density of the form

$$S_\chi(f) \equiv \int_{-\infty}^{\infty} d\tau\, R_\chi(\tau)\, e^{-j2\pi f\tau} = \frac{\sigma_\chi^2}{\pi W_\chi} \Big/ \left[1 + \left(\frac{f}{W_\chi}\right)^2\right] \tag{18}$$

Equation (18) shows that $\chi(t)$ has the required one-sided bandwidth W_x. Furthermore,

$$S_x(f)/S_x(0) \cong \left(\frac{f}{W_x}\right)^{-2}; \qquad |f| >> W_x \tag{19}$$

This high-performance asymptotic behavior conforms fairly well with Woo's theoretical analysis (Ref. 1), which shows that in fact

$$S_x(f)/S_x(0) \cong \left(\frac{f}{W_x}\right)^{-8/3}; \qquad |f| >> W_x \tag{20}$$

Using Eq. (17), and applying the results of Appendices B and C to the random variable α in Eq. (5), it follows that

$$\begin{aligned} \overline{\alpha} &= \exp\left(-\frac{\sigma_x^2}{2}\right) \\ \overline{\alpha^2} &\cong \exp(-\sigma_x^2)\left[1 + \frac{2\sigma_x^2}{\beta^2}(e^{-\beta} - 1 + \beta)\right]; \quad \sigma_x^2 << 1 \end{aligned} \tag{21}$$

where β is defined in Eq. (14).

Note that for $\beta << 1$ (or $R_B >> 1$), α becomes a lognormal random variable as in Eq. (9). At the other extreme, when $\beta >> 1$, the integral expression for α in Eq. (5) can be approximated by a sum:

$$\alpha \approx \frac{1}{N}\sum_{i=1}^{N} e^{\chi_i} \tag{22}$$

where the χ_i's are identically distributed, statistically independent Gaussian random variables, and N is the number of degrees of freedom of $\chi(t)$ over a baud time T_B:

$$N = \frac{T_B}{\left(\frac{1}{W_x}\right)} = \frac{\beta}{2\pi} \tag{23}$$

One might consider applying the Central Limit Theorem to Eq. (22) to conclude that α becomes Gaussian for large N (or β). But the probability density functions of the lognormal random variables e^{χ_i} are characterized by long tails, which makes the Gaussian approximation inaccurate, except near $\overline{\alpha}$. A better approximation for the probability density function of α, which is accurate farther into its tail, is the lognormal density. Mitchell (Ref. 8) has demonstrated that the sum of N statistically independent, identically distributed lognormal random variables may be accurately approximated by a lognormal random variable for large N.

Since α looks lognormal in the limits as $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, it will be assumed that α is approximately lognormal over the entire range of β:

$$\alpha \cong e^{\gamma} \tag{24}$$

where γ is a Gaussian random variable with mean m_γ and variance σ_γ^2. Then

$$\begin{aligned} \overline{\alpha} &= \exp\left(m_\gamma + \frac{\sigma_\gamma^2}{2}\right) \\ \overline{\alpha^2} &= \exp(2m_\gamma + 2\sigma_\gamma^2) \end{aligned} \tag{25}$$

Comparing Eqs. (21) and (25), it follows that

$$\begin{aligned} \sigma_\gamma^2 &= \ln\left[1 + \frac{2\sigma_x^2}{\beta^2}(e^{-\beta} - 1 + \beta)\right] \\ m_\gamma &= -\frac{1}{2}(\sigma_x^2 + \sigma_\gamma^2) \end{aligned} \tag{26}$$

In Fig. 2, m_γ/m_x ($m_x = -\sigma_x^2$) and $\sigma_\gamma^2/\sigma_x^2$ are plotted vs β, for $\sigma_x^2 = 0.014$. These curves show that for small β, the probability density function for γ is a broad Gaussian curve, with standard deviation σ_x, centered at $-\sigma_x^2$; as β increases, there is a smooth trend wherein the mean of γ shifts towards $-\sigma_x^2/2$ while the standard deviation drops to zero. For very large β, the probability density function of γ is essentially a Dirac-delta function centered at $-\sigma_x^2/2$.

Using this model, the expected bit error rate is given by the formula

$$P(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\gamma^2}}\int_{-\infty}^{\infty} d\gamma \exp - \left[\frac{(\gamma - m_\gamma)^2}{2\sigma_\gamma^2}\right] Q(e^{\gamma}\sqrt{2\rho}) \tag{27}$$

Applying numerical integration techniques to Eqs. (14), (26), and (27), we computed $P(\epsilon)$ as a function of ρ for $\sigma_x^2 = 0.014$, $W_x = 1$ Hz, and various values of R_B. The resulting bit error rate curves for $R_B = 16$ and 256 bps (the lowest and highest data rates currently being considered for Pioneer Venus 1978) are compared with the nonfading case in Fig. 3. (The bit error rate curves for $R_B = 64$ and 128 bps, the other two rates under consideration for the mission, lie too close to the $R_B = 256$ bps curve to be distinguished from it.) Comparing Figs. 1 and 3, it is evident that the bit error rate curves for the Pioneer Venus data rates, predicted by the intermediate rate model, lie close to the upper bound ($\beta = 0$) derived in the previous section. The deviation in ρ between the fading and non-fading curves in Fig. 3, at a given bit error probability $P(\epsilon)$, is the fading loss entry that would appear in the data channel section of a corresponding uncoded telemetry link design control table. In particular, for $\sigma_X^2 = 0.014$, $W_X = 1$ Hz, and $R_B = 256$ bps, the intermediate rate model predicts a fading loss of 0.6 dB at a bit error rate of 10^{-3}, and 1.0 dB at 10^{-5}.

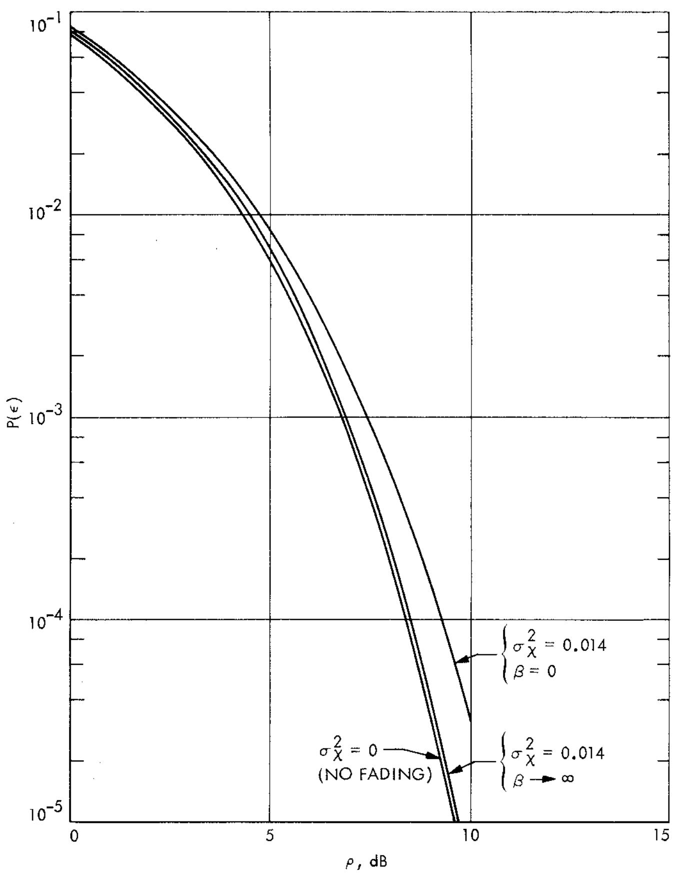

Fig. 1. Upper and lower bounds on the bit error rate for uncoded data received over a lognormal fading channel

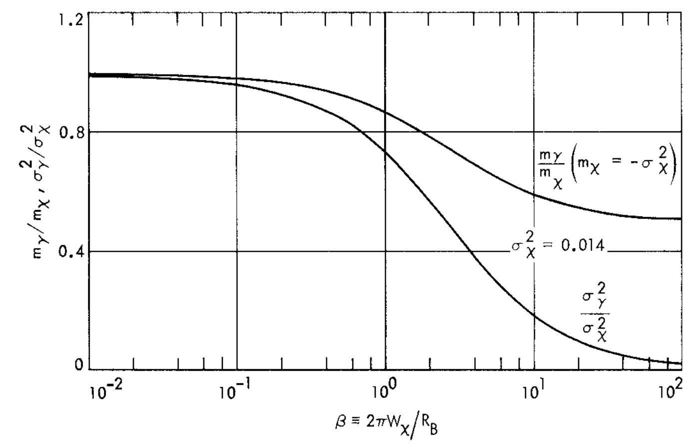

Fig. 2. Variation of intermediate rate model fading parameters with data rate $R_\beta = 2\pi W_X/\beta$

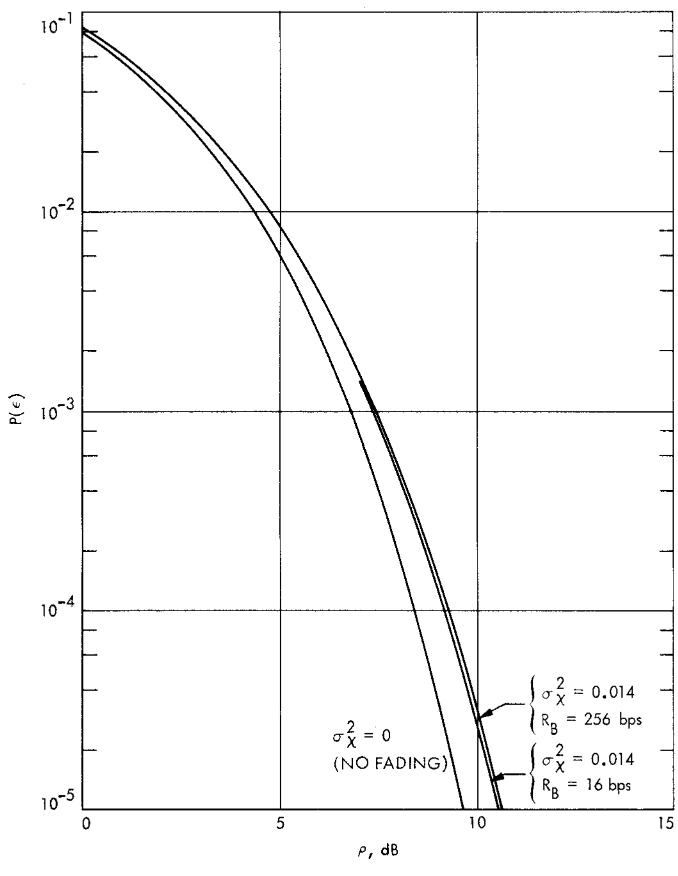

Fig. 3. Bit error rate performance of Pioneer Venus uncoded telemetry links, predicted by intermediate rate model

Appendix A

Lemma: Bounds on the Bit Error Rate for Uncoded Binary Data Received Over a Fading Gaussian Channel

In this section we will prove a general lemma developed by Dr. E. R. Rodemich of the Communications Systems Research Section.

LEMMA

Consider the random variable

$$\alpha = \frac{1}{T}\int_0^T dt\, x(t) \tag{A.1}$$

which is a T-second time average of the positive, stationary random process $x(t)$. Define the parameter

$$\epsilon = \overline{Q(k\alpha)} \tag{A.2}$$

where k is positive and fixed, $Q(\cdot)$ is the normalized Gaussian error function defined by

$$Q(\gamma) \equiv \frac{1}{\sqrt{2\pi}}\int_\gamma^\infty d\beta\, e^{-\beta^2/2} \tag{A.3}$$

and the overbar in Eq. (A.2) indicates the expectation of $Q(k\alpha)$. Then ϵ is bounded by

$$Q(k\bar{x}) \le \epsilon \le \overline{Q(kx)} \tag{A.4}$$

where the random variable $x \equiv x(t_0)$ for any fixed t_0.

Proof

For arbitrary integer n, define

$$\alpha_j \equiv \frac{n}{T}\int_{(j-1)(T/n)}^{j(T/n)} dt\, x(t); \qquad j = 1, 2, \cdots, n \tag{A.5}$$

so that

$$\alpha = \frac{1}{n}\sum_{j=1}^{n} \alpha_j \tag{A.6}$$

The α_j's are positive, identically distributed, (correlated) random variables. For positive γ, $Q(\gamma)$ is concave, since Eq. (A.3) implies that

$$\frac{d^2Q}{d\gamma^2} = \frac{1}{\sqrt{2\pi}}\gamma e^{-\gamma^2/2} \tag{A.7}$$

Therefore, applying Jensen's inequality for concave functions,

$$Q(k\alpha) \le \frac{1}{n}\sum_{j=1}^{n} Q(k\alpha_j) \tag{A.8}$$

$$\epsilon = \overline{Q(k\alpha)} \le \frac{1}{n}\sum_{j=1}^{n}\overline{Q(k\alpha_j)} = \overline{Q(k\alpha_j)}, \quad \text{for any } n, j \tag{A.9}$$

In particular,

$$\epsilon \le \overline{Q(k\alpha^*)} \tag{A.10}$$

where

$$\alpha^* \equiv \lim_{n\to\infty} \alpha_j; \qquad \text{for any } j \tag{A.11}$$

But α^* has the same probability distribution as x, so that

$$\epsilon \le \overline{Q(kx)} \tag{A.12}$$

Now define a new set of α_j's according to

$$\alpha_j \equiv \frac{1}{T}\int_{(j-1)T}^{jT} dt\, x(t); \qquad j = 1, 2, \cdots, n \tag{A.13}$$

so that $\alpha_1 = \alpha$. These α_j's are also positive, identically distributed random variables. Define

$$\alpha' \equiv \frac{1}{nT}\int_0^{nT} dt\, x(t) = \frac{1}{n}\sum_{j=1}^{n}\alpha_j \tag{A.14}$$

Again, using Jensen's inequality, it follows that

$$\overline{Q(k\alpha')} \le \frac{1}{n}\sum_{j=1}^{n}\overline{Q(k\alpha_j)} = \overline{Q(k\alpha_1)} = \epsilon, \qquad \text{for any } n \tag{A.15}$$

In particular,

$$\epsilon \geq \overline{Q(k\alpha^*)} \tag{A.16}$$

where we have redefined the random variable

$$\alpha^* \equiv \lim_{n\to\infty} \alpha' \tag{A.17}$$

But if $x(t)$ is an ergodic random process,

$$\alpha^* = \lim_{n\to\infty} \left[\frac{1}{nT} \int_0^{nT} dt\, x(t) \right] = \overline{x} \tag{A.18}$$

so that

$$\epsilon \geq Q(k\overline{x}) \qquad \text{(Q.E.D.)} \tag{A.19}$$

Appendix B

Crosscorrelation of Two Lognormal Random Variables

Suppose x_1 and x_2 are identically distributed, jointly Gaussian random variables, each with mean m and variance σ^2. The joint characteristic function of x_1 and x_2 is given by (Ref. 4, p. 163)

$$\begin{aligned} M_{x_1,x_2}(\nu_1,\nu_2) &\equiv \overline{\exp\left(j(\nu_1 x_1 + \nu_2 x_2)\right)} \\ &= \exp\left[jm(\nu_1+\nu_2) - \rho\nu_1\nu_2 - \frac{\sigma^2}{2}(\nu_1^2+\nu_2^2)\right] \end{aligned} \tag{B.1}$$

where ρ is the covariance of x_1 and x_2 defined by

$$\rho \equiv \overline{(x_1 - \overline{x_1})(x_2 - \overline{x_2})} \tag{B.2}$$

In particular, if $m = -\sigma^2$ and $\nu_1 = \nu_2 = -j$, Eq. (B.1) yields the result

$$\overline{e^{x_1}e^{x_2}} = \exp(\rho - \sigma^2) \tag{B.3}$$

Appendix C

Computation of First Two Moments of Lognormal Amplitude Fading Parameter α

Suppose $x(t)$ is a stationary, Gaussian random process with mean m equal to the negative of its variance σ^2. Suppose further that its covariance function is specified by

$$R_x(\tau) = \overline{[x(t+\tau) - \overline{x(t+\tau)}][x(t) - \overline{x(t)}]} = \sigma^2 e^{-2\pi W|\tau|} \tag{C.1}$$

where W is the approximate one-sided spectral bandwidth of $x(t)$. We would like to compute the first two moments of the random variable

$$\alpha = \frac{1}{T}\int_0^T dt\, e^{x(t)} \tag{C.2}$$

Clearly,

$$\overline{\alpha} = \overline{e^{x(t)}} = e^{\frac{-\sigma^2}{2}} \tag{C.3}$$

Also,

$$\overline{\alpha^2} = \frac{1}{T^2}\int_0^T dt \int_0^T d\mu\, \overline{\exp[x(t)]\exp[x(\mu)]}$$

$$= \frac{e^{-\sigma^2}}{T^2}\int_0^T dt \int_0^T d\mu\, e^{\sigma^2} e^{-2\pi W|t-\mu|} \tag{C.4}$$

where we have applied Eqs. (B.3) and (C.1). Defining the parameter

$$\beta \equiv 2\pi WT \tag{C.5}$$

and making the variable change

$$\eta = \frac{t}{T}$$

$$\zeta = \frac{t-\mu}{T} \tag{C.6}$$

the double integral of Eq. (C.4) becomes

$$\overline{\alpha^2} = e^{-\sigma^2}\int_0^1 d\eta \int_{\eta-1}^{\eta} d\zeta \exp[\sigma^2 \exp(-\beta|\zeta|)] \tag{C.7}$$

Equation (C.7) cannot in general be solved explicitly; however, for small σ^2 we can write

$$\overline{\alpha^2} \cong e^{-\sigma^2}\int_0^1 d\eta \int_{\eta-1}^{\eta} d\zeta\,(1 + \sigma^2 e^{-\beta|\zeta|}); \qquad \sigma^2 << 1 \tag{C.8}$$

$$\Downarrow$$

$$\overline{\alpha^2} \cong e^{-\sigma^2}\left[1 + \frac{2\sigma^2}{\beta^2}(e^{-\beta} - 1 + \beta)\right]; \qquad \sigma^2 << 1 \tag{C.9}$$

As a check on Eq. (C.9), note that

$$\overline{\alpha^2} \xrightarrow[\beta \to 0]{} \exp(-\sigma^2)(1+\sigma^2) \cong 1; \qquad \sigma^2 << 1$$

$$\overline{\alpha^2} \xrightarrow[\beta \to \infty]{} \exp(-\sigma^2) \tag{C.10}$$

But from the definition of α in Eq. (C.2), we can write

$$\alpha \xrightarrow[T \to 0]{} e^{x(t)}$$

$$\alpha \xrightarrow[T \to \infty]{} \overline{e^{x(t)}} = \exp\left(-\frac{\sigma^2}{2}\right) \tag{C.11}$$

$$\Downarrow$$

$$\overline{\alpha^2} \xrightarrow[T \to 0]{} \overline{e^{2x(t)}} = 1$$

$$\overline{\alpha^2} \xrightarrow[T \to \infty]{} \exp(-\sigma^2) \tag{C.12}$$

which confirms Eq. (C.10).

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